Quantum wire junctions breaking time-reversal invariance

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We explore the possibility to break time-reversal invariance at the junction of quantum wires. The universal features in the bulk of the wires are described by the anyon Luttinger liquid. A simple necessary and sufficient condition for the breaking of time-reversal invariance is formulated in terms of the scattering matrix at the junction. The phase diagram of a junction with generic number of wires is investigated in this framework. We give an explicit classification of those critical points, which can be reached by bosonization and study the interplay between their stability and symmetry content.

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I. INTRODUCTION

Time-reversal symmetry is a fascinating subject. In this paper we investigate the behavior of junctions of quantum wires under time-reversal transformations. Quantum wire networks with junctions, which attract recently much attention,[1–29] are essentially one-dimensional systems whose transport properties are affected by quantum effects. The universal features in the bulk are captured by the Luttinger liquid theory.[30] The junctions represent in this context a kind of quantum impurities (defects), where both reflection and transmission can take place. This fact gives origin of a complicated phase diagram, which has not been yet fully understood for general boundary conditions at the junctions, formulated in terms of the basic fermion fields. Focusing on the case of one junction, we discovered[19] in the framework of bosonization a large class of boundary conditions, which preserve the exact solvability of the Tomonaga-Luttinger (TL) model describing the Luttinger liquid in the bulk. At criticality these boundary conditions simply express the splitting of the electric current in the junction and are therefore quadratic in the fermion fields. We classified and studied in this setting all critical points which respect time-reversal invariance. In this paper we extend our framework in order to cover also that part of the phase diagram, where the time-reversal symmetry is broken. Recalling that the Tomonaga-Luttinger dynamics preserves time-reversal invariance, the breaking can take place only at the junctions. In principle such kind of junctions can be realized[9,10,15,24] by means of an external magnetic field and are therefore of practical interest.

The previous theoretical investigations of the stability of the critical points and their behavior under time reversal have been mostly focused on junctions with $n=3,4$ wires. Applying the framework developed in Refs. 19–21, we face below these problems for generic n.

The paper is organized as follows. In the next section we define the bulk dynamics and boundary conditions at the junction. Using bosonization, we recall[21] in Sec. III the exact (anyon) solution of the model. In Sec. IV we derive the current-current correlation function and extract the necessary and sufficient condition for the breaking of time reversal. We discuss here also the Kirchhoff's rules relative to the $U(1) \otimes \widetilde{U}(1)$ symmetry of the model. In Sec. V we consider the conductance and describe the impact of time-reversal breaking on it. The classification and parametrization of the critical points is done in Sec. VI. In Sec. VII we study the phase diagram, concentrating mainly on the symmetry content and stability of the fixed points. Section VIII is devoted to our conclusions. Some technical details are collected in the appendixes.

II. BULK DYNAMICS, SYMMETRIES, AND BOUNDARY CONDITIONS

The quantum wire junction is modeled by a star graph Γ of the form shown in Fig. 1. The edges E_i are half lines and each point P in the bulk of Γ is uniquely determined by its coordinates (x,i), where $x>0$ is the distance to the vertex V and $i=1,\ldots,n$ labels the edge. $\Gamma \setminus V$ represents the *bulk* of the graph. The bulk dynamics is governed by the TL Lagrangian density

$$\mathcal{L} = i\psi_1^*(\partial_t - v_F\partial_x)\psi_1 + i\psi_2^*(\partial_t + v_F\partial_x)\psi_2 - g_+(\psi_1^*\psi_1 + \psi_2^*\psi_2)^2 - g_-(\psi_1^*\psi_1 - \psi_2^*\psi_2)^2. \quad (1)$$

Here, $\{\psi_\alpha(t,x,i):\alpha=1,2\}$ are complex fields, v_F is the Fermi

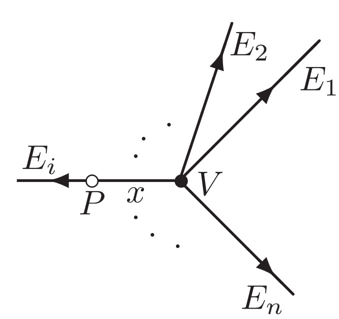

FIG. 1. A star graph Γ with n edges modeling the junction of quantum wires.

velocity, and $g_{\pm} \in \mathbb{R}$ are the coupling constants.[31]

The bulk theory has an obvious $U(1)\otimes\widetilde{U}(1)$ symmetry. In fact, the Lagrangian density (1) is left invariant by the two independent phase transformations $(s,\widetilde{s}\in\mathbb{R})$,

$$\psi_\alpha \to e^{is}\psi_\alpha, \quad \psi^*_\alpha \to e^{-is}\psi^*_\alpha, \tag{2}$$

$$\psi_\alpha \to e^{-i(-1)^\alpha \widetilde{s}}\psi_\alpha, \quad \psi^*_\alpha \to e^{i(-1)^\alpha \widetilde{s}}\psi^*_\alpha, \tag{3}$$

implying the current conservation laws

$$\partial_t \rho_\pm(t,x,i) - v_F \partial_x j_\pm(t,x,i) = 0, \tag{4}$$

where

$$\rho_\pm(t,x,i) = (\psi^*_1\psi_1 \pm \psi^*_2\psi_2)(t,x,i) \tag{5}$$

are the charge densities and

$$j_\pm(t,x,i) = \rho_\mp(t,x,i) \tag{6}$$

are relative currents. We adopt below also the chiral combinations

$$j_R(t,x,i) = \tfrac{1}{2}(\zeta_- j_- + \zeta_+ j_+)(t,x,i), \tag{7}$$

$$j_L(t,x,i) = \tfrac{1}{2}(\zeta_- j_- - \zeta_+ j_+)(t,x,i), \tag{8}$$

where the real parameters $\zeta_\pm$, determined later on, are such that j_L and j_R represent the particle excitations moving toward and away of the vertex, respectively. Interpreting the vertex as a defect, characterized by some scattering matrix, the currents j_L and j_R describe therefore the incoming and outgoing flows.

The bulk theory is invariant also under the time-reversal operation,

$$T\psi_1(t,x,i)T^* = \psi_2(-t,x,i), \tag{9}$$

$$T\psi_2(t,x,i)T^* = \psi_1(-t,x,i), \tag{10}$$

where T is an *antiunitary* operator. As already mentioned, our main goal below will be to investigate the impact of the vertex V and the related boundary conditions on time reversal.

The TL model (1) is exactly solvable on the line $\mathbb{R}$, but much care is needed on the graph Γ, where some boundary conditions must be imposed at the vertex V. Keeping in mind that the quartic bulk interactions in Eq. (1) can be solved exactly via bosonization,[30] it will be obviously convenient to formulate the boundary conditions directly in bosonic terms. In this spirit and according our previous comments on the chiral currents, it is quite natural to require that at a critical point

$$j_L(t,0,i) = \sum_{k=1}^{n} \mathbb{S}_{ik} j_R(t,0,k), \quad \forall\, t \in \mathbb{R}. \tag{11}$$

For $n=2$ this boundary condition has been first proposed and explored in Ref. 2. Because of scale invariance at criticality, $\mathbb{S}$ is a constant (momentum independent) unitary scattering matrix,

$$\mathbb{S}\mathbb{S}^* = \mathbb{I}. \tag{12}$$

Since the chiral currents (7) and (8) are Hermitian fields, one requires also that $\mathbb{S}$ has real entries,

$$\overline{\mathbb{S}} = \mathbb{S}. \tag{13}$$

Equations (12) and (13) imply that $\mathbb{S}$ is any element of the orthogonal group $O(n)$. It has been shown in Refs. 19–21 that, in spite of the fact that the boundary condition (11) is quadratic in the fields ψ_α, it preserves the exact solvability of the TL model on the graph Γ. It is worth mentioning that this is not the case with the linear boundary conditions in ψ_α, which might look at first sight simpler.

Applying the time-reversal operations (9) and (10) to Eq. (11) one infers that

$$j_L(t,0,i) = \sum_{k=1}^{n} \mathbb{S}^t_{ik} j_R(t,0,k), \quad \forall\, t \in \mathbb{R}, \tag{14}$$

where the apex t stands for transposition. Comparing Eqs. (11) and (14) we conclude that *symmetric* scattering matrices

$$\mathbb{S} = \mathbb{S}^t \tag{15}$$

define boundary conditions which respect the time-reversal invariance. This is the case we investigated previously in Refs. 19–21. On the other hand, for

$$\mathbb{S} \neq \mathbb{S}^t \tag{16}$$

one expects breaking of time reversal. We demonstrate in Sec. IV that this is indeed the case, using the explicit form of the current-current correlation function. We conclude at this point the concise description of the bulk dynamics, symmetries, and boundary conditions of our model and briefly describe in the next section the solution.

III. SOLUTION OF THE TL MODEL ON A STAR GRAPH

We look below for the solution ψ_α of the TL model which satisfies the boundary condition (11) and obeys the anyon exchange relations

$$\psi^*_\alpha(t,x_1,i)\psi_\alpha(t,x_2,i) = e^{(-1)^\alpha i\pi\kappa\varepsilon(x_{12})}\psi_\alpha(t,x_2,i)\psi^*_\alpha(t,x_1,i). \tag{17}$$

Here, $\varepsilon(x)$ is the sign function, $x_{12}=x_1-x_2$, and $\kappa\in\mathbb{R}$ is the so called *statistical parameter*, which equals an even and an odd integer for bosons and fermions, respectively. Other values of κ give rise to Abelian anyon statistics "interpolating" between bosons and fermions.

The solution on Γ can be expressed in terms of the chiral scalar fields $\{\varphi_{i,Z}(\xi): Z=L,R; i=1,\ldots,n\}$, which are not independent as on the line $\mathbb{R}$, but respect the constraints

$$\varphi_{i,L}(\xi) = \sum_{j=1}^{n} \mathbb{S}_{ij}\varphi_{j,R}(\xi), \tag{18}$$

keeping track of the boundary conditions (11). The explicit construction and a summary of the main features of $\varphi_{i,Z}$ are given in Appendix A. A key point is the nontrivial one-body scattering matrix[32]

$$S(k) = \theta(-k)\mathbb{S} + \theta(k)\mathbb{S}^t, \tag{19}$$

where θ is the Heaviside step function. We stress that the peculiar k dependence of $S(k)$ respects scale invariance.

Let us summarize now the basic features of the solution of the TL model with boundary conditions (11). We do this essentially for two reasons. First of all the field φ associated with the S matrix (19) behaves quite differently (see Appendix A) from its counterpart in Ref. 21. Second, because we would like to keep the present paper self-contained.

Following the standard bosonization procedure,[30] we set

$$\psi_1(t,x,i) = z_i {:} e^{i\sqrt{\pi}[\sigma\varphi_{i,R}(vt-x)+\tau\varphi_{i,L}(vt+x)]}{:}, \tag{20}$$

$$\psi_2(t,x,i) = z_i {:} e^{i\sqrt{\pi}[\tau\varphi_{i,R}(vt-x)+\sigma\varphi_{i,L}(vt+x)]}{:}, \tag{21}$$

where $:\cdots:$ denotes the normal product relative to the creation and annihilation operators of the fields $\varphi_{i,Z}$, namely, the generators of the algebra (A2). The explicit form of the normalization constants z_i (including the Klein factors) is reported in Appendix A as well. Finally, σ, τ, and v are three real parameters to be determined in terms of coupling constants $g_\pm$ and the statistical parameter κ. For this purpose we can assume without loss of generality that

$$\sigma \geq 0, \quad \sigma \neq \pm\tau, \tag{22}$$

and introduce for convenience the variables

$$\zeta_\pm = \tau \pm \sigma. \tag{23}$$

Plugging Eqs. (20) and (21) in Eq. (17) one gets

$$\zeta_+\zeta_- = \kappa. \tag{24}$$

Moreover, using standard short-distance expansion for the charge densities, one obtains

$$\rho_\pm(t,x,i) = \frac{-1}{2\sqrt{\pi}\zeta_\pm}[(\partial\varphi_{i,R})(vt-x) \pm (\partial\varphi_{i,L})(vt+x)], \tag{25}$$

with the normalization being fixed[21] by the $U(1)\otimes\widetilde{U}(1)$-Ward identities. Inserting Eqs. (20), (21), and (25) in the quantum equations of motion,

$$\begin{aligned} &i[\partial_t + (-1)^\alpha v_F\partial_x]\psi_\alpha(t,x,i) \\ &\quad = 2[g_+{:}\rho_+(t,x,i)\psi_\alpha{:}(t,x,i) - (-1)^\alpha g_-{:}\rho_-(t,x,i)\psi_\alpha{:}(t,x,i)], \end{aligned} \tag{26}$$

one finds

$$v\zeta_+^2 = v_F\kappa + \frac{2}{\pi}g_+, \tag{27}$$

$$v\zeta_-^2 = v_F\kappa + \frac{2}{\pi}g_-. \tag{28}$$

Equations (24), (27), and (28) provide a system for determining v and $\zeta_\pm$ (equivalently σ and τ) in terms of v_F and $g_\pm$. The solution is

$$\zeta_\pm^2 = |\kappa|\left(\frac{\pi\kappa v_F + 2g_+}{\pi\kappa v_F + 2g_-}\right)^{\pm 1/2}, \tag{29}$$

$$v = \frac{\sqrt{(\pi\kappa v_F + 2g_-)(\pi\kappa v_F + 2g_+)}}{\pi|\kappa|}, \tag{30}$$

where the positive roots are taken in the right-hand side. Relations (29) and (30) represent the anyonic generalization[21] of the well-known result for canonical fermions ($\kappa=1$) in the TL model.[33] The conditions $2g_\pm > -\pi\kappa v_F$ ensure that σ, τ, and v are real and finite.

Finally, in the bosonic variables $U(1)\otimes\widetilde{U}(1)$ currents $j_\pm$ take the form

$$j_\pm(t,x,i) = \frac{v}{2\sqrt{\pi}v_F\zeta_\pm}[(\partial\varphi_{i,R})(vt-x) \mp (\partial\varphi_{i,L})(vt+x)] \tag{31}$$

and satisfy Eq. (4) by construction. Using Eqs. (18) and (31) one immediately verifies that the above solution of the TL model on Γ indeed satisfies the boundary condition (11).

IV. SYMMETRY CONTENT

A. Time reversal

The simplest way to investigate the behavior of the above solution under time reversal is to derive the two-point correlation functions of the currents $j_\pm$, defined by Eq. (31). Using Eqs. (A10)–(A12) one obtains

$$\begin{aligned} &\langle j_+(t_1,x_1,i_1)j_+(t_2,x_2,i_2)\rangle \\ &\quad = \frac{v^2}{(2\pi\zeta_+v_F)^2}[\delta_{i_1i_2}\mathcal{D}^2(vt_{12}-x_{12}) + \delta_{i_1i_2}\mathcal{D}^2(vt_{12}+x_{12}) \\ &\qquad - \mathbb{S}_{i_1i_2}\mathcal{D}^2(vt_{12}+\widetilde{x}_{12}) - \mathbb{S}^t_{i_1i_2}\mathcal{D}^2(vt_{12}-\widetilde{x}_{12})], \end{aligned} \tag{32}$$

where

$$\mathcal{D}(\xi) = -\frac{i}{\xi + i\epsilon}, \tag{33}$$

and $t_{12}=t_1-t_2$, $x_{12}=x_1-x_2$, and $\widetilde{x}_{12}=x_1+x_2$.

Let us assume for a moment that time reversal is an exact symmetry or, equivalently, that T leaves invariant the vacuum state Ω. Then, using the antiunitarity of T, one finds that

$$\langle j_+(t_1,x_1,i_1)j_+(t_2,x_2,i_2)\rangle = \overline{\langle j_+(-t_1,x_1,i_1)j_+(-t_2,x_2,i_2)\rangle} \tag{34}$$

holds. Combining Eq. (32) with Eq. (34) one deduces that

$$T\Omega = \Omega \Leftrightarrow \mathbb{S} = \mathbb{S}^t, \tag{35}$$

showing that the TL model on Γ is invariant under time reversal if and only if $\mathbb{S}$ is symmetric. Otherwise, time reversal is broken, i.e.,

$$T\Omega \neq \Omega \Leftrightarrow \mathbb{S} \neq \mathbb{S}^t, \tag{36}$$

which confirms the conjecture made after Eq. (16) in Sec. II. In particular, time reversal is always exact for $n=1$. For this reason we focus in what follows on the case $n\geq 2$.

B. $U(1)\otimes\tilde{U}(1)$ symmetry

Continuous symmetries on graphs are governed[19,20] by the associated Kirchhoff's rules. Concerning the $U(1)\otimes\tilde{U}(1)$ symmetry, using the current conservation (4), one gets for the corresponding charges

$$\partial_t Q_\pm = \partial_t \sum_{i=1}^n \int_0^\infty dx\, \rho_\pm(t,x,i) = v_F \sum_{i=1}^n j_\pm(t,0,i). \tag{37}$$

Inserting here Eq. (31) and taking into account the boundary conditions (18), one finds

$$\partial_t Q_\pm = \frac{v}{2\sqrt{\pi}\zeta_\pm} \sum_{i,j=1}^n (\delta_{ij} \mp \mathbb{S}_{ij})(\partial\varphi_{j,R})(vt). \tag{38}$$

From this result we infer that[19]

$$Q_\pm - \text{conserved} \Leftrightarrow \sum_{i=1}^n \mathbb{S}_{ij} = \pm 1, \quad \forall\, j = 1, \ldots, n. \tag{39}$$

Recalling that Q_+ is the electric charge and Q_- is the helicity of the Luttinger excitations, we see that only one of these quantum numbers is preserved for a generic junction.[34]

It is worth mentioning that for junctions with $n=2$ wires the conservation of Q_+ or Q_- protects the time-reversal symmetry. Thus, junctions of three wires (T junctions and Y junctions) represent the minimal setting for breaking time reversal in systems preserving the electric charge Q_+ or the helicity Q_- of the Luttinger liquid. Notice also that the conservation of Q_+ excludes the Dirichlet fixed point $\mathbb{S}=-\mathbb{I}$.

V. CONDUCTANCE

A simple physical observable, which is sensitive to the breaking of time reversal, is the conductance tensor $\mathbb{G}_{ij}$ of the Luttinger liquid on Γ. In order to compute this tensor, one couples the theory to an external potential $A_x(t,i)$ by means of the substitution

$$\partial_x \mapsto \partial_x + \mathrm{i}A_x(t,i) \tag{40}$$

in Eq. (1). The resulting Hamiltonian is *time-dependent* and the conductance is the coefficient in the linear term of the expansion of the expectation value $\langle J_x(t,0,i)\rangle_{A_x}$ in terms of A_x. For deriving $\mathbb{G}$ one can apply therefore linear-response theory, which leads to[19,20]

$$\mathbb{G} = \frac{v}{2\pi v_F \zeta_+^2}(\mathbb{I} - \mathbb{S}). \tag{41}$$

Using condition (39), which ensures the conservation of the electric charge Q_+ and $\mathbb{S}\in O(n)$, one gets the Kirchhoff's rule for the conductance tensor,

$$\sum_{i=1}^n \mathbb{G}_{ij} = \sum_{i=1}^n \mathbb{G}_{ji} = 0, \quad \forall\, j = 1, \ldots, n. \tag{42}$$

If a voltage V_i is applied to the edge E_i, the current I_j flowing in E_j is

$$I_j = \sum_{i=1}^n \mathbb{G}_{ji} V_i. \tag{43}$$

Combining Eq. (16) with Eq. (41) we conclude that the breaking of time reversal (16) implies the asymmetry

$$\mathbb{G} \neq \mathbb{G}^t, \tag{44}$$

a feature which has been previously observed in Refs. 7, 10, and 22. Property (44) provides an attractive experimental signature. Indeed, consider, for instance, the following two configurations with $i\neq j$. Apply first the voltage V to the edge E_i, setting to zero the voltages in all other edges and measure the current $I_j=\mathbb{G}_{ji}V$. Repeat the same operation, applying now V to the edge E_j and measuring the current $I_i=\mathbb{G}_{ij}V$. If $I_i/I_j\neq 1$, the system breaks time reversal.

VI. CRITICAL POINTS

A. Classification

As already mentioned, $\mathbb{S}\in O(n)$. There exists therefore an orthogonal matrix $\mathbb{O}$, such that

$$\mathbb{O}\mathbb{S}\mathbb{O}^t = \begin{pmatrix} r_1 & & & & & \\ & \ddots & & & 0 & \\ & & r_q & & & \\ & & & \pm 1 & & \\ & 0 & & & \ddots & \\ & & & & & \pm 1 \end{pmatrix}. \tag{45}$$

Here, r_i are q rotation matrices

$$r_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}, \quad \theta_i \in [-\pi,\pi). \tag{46}$$

Let us denote by $p_\pm$ the number of eigenvalues ± 1 of Eq. (46). Then $q=\frac{1}{2}(n-p_+-p_-)$ and the critical points are classified by the set $(p_+,p_-,\theta_1,\ldots,\theta_q)$. From Eq. (36) we conclude that the time-reversal symmetry is broken if and only if $\theta_k\neq -\pi,0$ for some $k=1,\ldots,q$. The angles θ_k thus codify the breaking of time reversal.

B. Parametrization

$\mathbb{S}$ can be any element of $O(n)$, but in the physical applications one is mostly interested in boundary conditions which preserve the electric charge Q_+. In this case one infers from Eq. (39) that

$$\mathbb{S}\mathbf{v} = \mathbf{v}, \quad \mathbf{v} = \frac{1}{\sqrt{n}}(1,1,\ldots,1), \tag{47}$$

i.e., $\mathbb{S}$ leaves invariant the vector $\mathbf{v}$. Let R be the orthogonal matrix [see Eq. (B2) in Appendix B], which rotates the vector $(0,0,\ldots,0,1)$ in $\mathbf{v}$. Then $\mathbb{S}$ admits the representation

$$\mathbb{S} = R\begin{pmatrix} \mathbb{S}' & 0 \\ 0 & 1 \end{pmatrix}R^t, \tag{48}$$

with $\mathbb{S}' \in O(n-1)$. Therefore, the boundary conditions which respect the $U(1)$ symmetry of the TL model are parametrized by the group $O(n-1)$. The two connected components of $O(n-1)$ give the origin of two continuous families of critical points. Each family depends on $(n-1)(n-2)/2$ parameters, which are the angular variables parametrizing the elements of $O(n-1)$.

Let us illustrate this simple general structure for $n=3$. The two families of critical points depend in this case on one angle $\vartheta \in [-\pi, \pi)$ and read

$$\mathbb{S}^{(1)}(\vartheta) = \frac{1}{3}\begin{pmatrix} 1+2\cos\vartheta & 1-\cos\vartheta+\sqrt{3}\sin\vartheta & 1-\cos\vartheta-\sqrt{3}\sin\vartheta \\ 1-\cos\vartheta-\sqrt{3}\sin\vartheta & 1+2\cos\vartheta & 1-\cos\vartheta+\sqrt{3}\sin\vartheta \\ 1-\cos\vartheta+\sqrt{3}\sin\vartheta & 1-\cos\vartheta-\sqrt{3}\sin\vartheta & 1+2\cos\vartheta \end{pmatrix}, \tag{49}$$

$$\mathbb{S}^{(2)}(\vartheta) = \frac{1}{3}\begin{pmatrix} 1-2\cos\vartheta & 1+\cos\vartheta-\sqrt{3}\sin\vartheta & 1+\cos\vartheta+\sqrt{3}\sin\vartheta \\ 1+\cos\vartheta-\sqrt{3}\sin\vartheta & 1+\cos\vartheta+\sqrt{3}\sin\vartheta & 1-2\cos\vartheta \\ 1+\cos\vartheta+\sqrt{3}\sin\vartheta & 1-2\cos\vartheta & 1+\cos\vartheta-\sqrt{3}\sin\vartheta \end{pmatrix}, \tag{50}$$

confirming the recent results of Ref. 26. The point

$$\mathbb{S}^{(1)}(-\pi) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \tag{51}$$

has been discovered by Griffith[35] more than five decades ago in his pioneering work on graph models in quantum chemistry. According to Eq. (41), in this case the conductance of the Luttinger liquid is enhanced with respect to the line, which has been associated[3] with the phenomenon of Andreev reflection. The Neumann point

$$\mathbb{S}^{(1)}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{52}$$

describes instead an ideal isolator because $\mathbb{G}=0$. To our knowledge the whole family $\mathbb{S}^{(2)}(\vartheta)$ was derived[36] first in Ref. 19 and, together with points (51) and (52), preserves time-reversal invariance. Finally, the matrices $S^{(1)}(\vartheta)$ with $\vartheta \neq -\pi, 0$ give all critical points which violate time-reversal symmetry for $n=3$. In a different parametrization[37] they appeared in Refs. 10 and 22.

In Appendix B we report an explicit parametrization of the $n\times n$ critical $\mathbb{S}$ matrices. The case when the $\widetilde{U}(1)$ symmetry is preserved can be treated analogously.[20]

VII. PHASE DIAGRAM: BOUNDARY DIMENSIONS AND STABILITY OF CRITICAL POINTS

The boundary dimensions of the solution ψ_α capture the impact of the junction at criticality and can be extracted from the two-point functions

$$\begin{aligned}\langle \psi_1^*(t_1,x_1,i_1)\psi_1(t_2,x_2,i_2)\rangle &= z_{i_1}z_{i_2}[\mathcal{D}(vt_{12}-x_{12})]^{\sigma^2\delta_{i_1i_2}}[\mathcal{D}(vt_{12}+x_{12})]^{\tau^2\delta_{i_1i_2}} \\ &\times[\mathcal{D}(vt_{12}-\tilde{x}_{12})]^{\sigma\tau\mathbb{S}^t_{i_1i_2}}[\mathcal{D}(vt_{12}+\tilde{x}_{12})]^{\sigma\tau\mathbb{S}_{i_1i_2}},\end{aligned} \tag{53}$$

and

$$\langle \psi_2^*(t_1,x_1,i_1)\psi_2(t_2,x_2,i_2)\rangle = (53), \quad \text{with } \sigma \leftrightarrow \tau. \tag{54}$$

Performing the scaling transformation

$$t \mapsto \varrho t, \quad x \mapsto \varrho x, \quad \varrho > 0 \tag{55}$$

in Eqs. (53) and (54), one obtains

$$\begin{aligned}&\langle \psi_\alpha^*(\varrho t_1,\varrho x_1,i_1)\psi_\alpha(\varrho t_2,\varrho x_2,i_2)\rangle \\ &= \varrho^{-D_{i_1i_2}}\langle \psi_\alpha^*(t_1,x_1,i_1)\psi_\alpha(t_2,x_2,i_2)\rangle,\end{aligned} \tag{56}$$

where

$$D = (\sigma^2+\tau^2)\mathbb{I}_n + \sigma\tau(\mathbb{S}+\mathbb{S}^t). \tag{57}$$

The eigenvalues d_i of the matrix $D/2$ are the scaling dimensions. If time reversal is broken $(\mathbb{S}\neq\mathbb{S}^t)$, some of the eigenvalues of $\mathbb{S}$ are necessarily complex. Notice however that the eigenvalues of the combination $\mathbb{S}+\mathbb{S}^t$ are all real and

$$d_i = \tfrac{1}{2}(\sigma^2+\tau^2) + \sigma\tau s_i, \tag{58}$$

with s being the n vector

$$s = (\cos\theta_1, \cos\theta_1, \cos\theta_2, \cos\theta_2, \ldots, \cos\theta_{2q}, \pm 1, \ldots, \pm 1). \tag{59}$$

In Appendix C we prove that the mixing between ψ_1 and ψ_2 produces vanishing additional eigenvalues and therefore does not affect the spectrum (58) and (59).

Recalling that the scaling dimension on the line is[38]

$$d^{(\text{line})} = \tfrac{1}{2}(\sigma^2+\tau^2), \tag{60}$$

one deduces from Eq. (58) the *boundary dimensions*

$$d_i^{(\text{boundary})} = \sigma\tau s_i = \frac{\zeta_+^2 - \zeta_-^2}{4} s_i, \qquad (61)$$

which control[38,39] the stability of the critical points. The direction i at a critical point $\mathbb{S}$ of the phase diagram is stable (unstable) if $d_i > 0$ ($d_i < 0$). We call point $\mathbb{S}$ *completely* stable if all relative directions are stable. Using Eqs. (27) and (28), the boundary dimension d_i can be rewritten in our case in the form

$$d_i^{(\text{boundary})} = \frac{1}{2\pi v}(g_+ - g_-) s_i, \quad v > 0, \qquad (62)$$

where $v > 0$ is given by Eq. (30). It is natural to consider at this point the two regimes of *repulsive* ($g_+ > g_-$) and *attractive* ($g_+ < g_-$) anyonic interactions. From Eq. (62) one concludes that in the repulsive case the direction i is stable if $s_i > 0$. Vice versa, the attractive case stability requires $s_i < 0$.

The direction i in the phase diagram is called flat if $d_i = 0$. This happens for $\cos\theta_i = 0$ and/or $g_+ = g_-$. The last case is very special: there is no interaction between ψ_1 and ψ_2 [see Eq. (1)]; all boundary dimensions vanish and all directions are flat.

It is worth stressing that the above considerations concern the phase diagram of the system without symmetry constraints. According to Sec. IV, however, the Kirchhoff's rules controlling the symmetry content of the TL model on Γ impose such constraints. If one requires for instance $U(1)$ symmetry, condition (39) implies that $s_i = 1$ in at least one direction. Therefore, for attractive interactions with $U(1)$ symmetry there are no completely stable points. The same conclusion holds in the repulsive case with $\tilde{U}(1)$ symmetry.

As already mentioned, imposing time-reversal symmetry implies that $\theta_i = -\pi$ or $\theta_i = 0$ for all $i = 1, \ldots, q$, which severely restricts the phase diagram. In particular, the only completely stable fixed points are $\mathbb{S} = \mathbb{I}$ (for $g_+ > g_-$) and $\mathbb{S} = -\mathbb{I}$ (for $g_+ < g_-$), corresponding to Neumann and Dirichlet boundary conditions, respectively. Allowing for breaking of time reversal leads to a richer phase diagram, which admits whole families of nontrivial ($\mathbb{S} \neq \pm\mathbb{I}$) completely stable critical points.

Let us consider for illustration the phase diagram for $n = 3$ (Y junction). We have shown above that all critical points, respecting the electric charge conservation, are given by Eqs. (49) and (50). The corresponding eigenvalues are

$$s^{(1)} = (\cos\theta, \cos\theta, 1), \quad s^{(2)} = (1, 1, -1), \qquad (63)$$

showing that the family $\mathbb{S}^{(2)}(\theta)$, which preserves time-reversal symmetry, does not contain completely stable points. The time-reversal breaking family $\mathbb{S}^{(1)}(\theta)$ contains instead the nontrivial completely stable fixed points with $\cos\theta > 0$ in the repulsive regime and $\cos\theta < 0$ in the attractive one. In this sense complete stability is favored by time-reversal breaking.

We stress in conclusion that the above algorithm can be applied for analyzing the stability of the critical points under perturbation with any composite operator involving the basic fields $\psi_\alpha(t, x, i)$. Some examples of quadratic operators are considered in Appendix D.

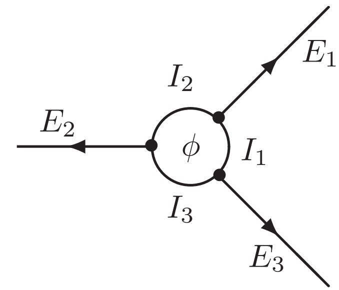

FIG. 2. A graph with three external and three internal edges.

VIII. CONCLUSIONS

We investigated above the behavior under time reversal of a Luttinger junction with any number of edges n and satisfying the boundary conditions (11). As expected, time-reversal invariance can be broken by boundary effects, in spite of the fact that the bulk theory preserves this symmetry. The following two exceptions are worth mentioning. Time-reversal symmetry is always preserved for $n = 1$. The same conclusion holds for $n = 2$, provided that the electric charge Q_+ is conserved.

The results of this paper give a global view on the phase diagram of the system with boundary conditions (11) and the framework allows us to investigate both the symmetry content and the stability of the critical points. It turns out that the phase diagram has two connected components, corresponding to those of the group $O(n)$ and therefore depending on $n(n-1)/2$ parameters, which describe irrelevant boundary couplings. In this classification the critical points, which respect the electric charge conservation, form a $O(n-1)$ subfamily. A simple criterion (16) allows us to distinguish the points which violate time-reversal invariance from those which preserve it. The stability of the critical points is controlled by the relative boundary dimensions. For generic n we derived these dimensions in explicit form (62), establishing their dependence on the boundary conditions and the bulk couplings. The analysis of the critical points, which are stable in all directions of the phase diagram, reveals that except for the Neumann point $\mathbb{S} = \mathbb{I}$ for repulsive interactions and the Dirichlet point $\mathbb{S} = -\mathbb{I}$ in the attractive case, all other completely stable points violate time-reversal invariance.

As already mentioned in the Introduction, the simplest realization of devices, violating time-reversal invariance, uses[9,10,15,24] magnetic fields. An example, which frequently appears in the literature,[10,22] is the configuration shown in Fig. 2. One has three external half lines and a ring composed of three compact internal edges and three junctions. A magnetic flux ϕ is crossing the ring. The complete field theory analysis of the Luttinger liquid on a graph with this geometry is very complicated problem, which is beyond the scope of the present paper. One approximate way to face the problem could be to use the star product approach[40] or the "gluing" technique[41,42] for deriving the 3×3 scattering matrix relative to the *external* edges. Although a bit complicated,[40,42,50] this S matrix can be used afterward for developing a simplified model with *one effective* junction. Clearly, such an approach does not provide the conductance of the internal edges I_i.

The generalization of the results of this paper to off-critical junctions represents also a challenging open problem. The study of the rich spectrum[29] of effects away of equilibrium is essential in this respect. Another interesting subject is the study of networks with several junctions. We are currently investigating these issues.

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APPENDIX A: CHIRAL FIELDS ON Γ

The chiral scalar fields,

$$\varphi_{i,R}(\xi) = \int_0^\infty \frac{dk}{\pi\sqrt{2k}}[a_i^*(k)e^{\mathrm{i}k\xi} + a_i(k)e^{-\mathrm{i}k\xi}],$$

$$\varphi_{i,L}(\xi) = \int_0^\infty \frac{dk}{\pi\sqrt{2k}}[a_i^*(-k)e^{\mathrm{i}k\xi} + a_i(-k)e^{-\mathrm{i}k\xi}], \quad \text{(A1)}$$

are the building blocks of the solution (20) and (21). On Γ the generators $\{a_i(k), a_i^*(k)\}$ obey the following *deformation*:

$$[a_i(k), a_j(p)] = [a_i^*(k), a_j^*(p)] = 0,$$

$$[a_i(k), a_j^*(p)] = 2\pi[\delta(k-p)\delta_{ij} + \delta(k+p)S_{ij}(k)], \quad \text{(A2)}$$

of the standard canonical commutation relations. Here, $S(k)$ is the one-body scattering matrix defined by Eq. (19). Besides Eq. (A2), we impose also the constraints

$$a_i(k) = \sum_{j=1}^n S_{ij}(k)a_j(-k), \quad \text{(A3)}$$

$$a_i^*(k) = \sum_{j=1}^n a_j^*(-k)S_{ji}(-k), \quad \text{(A4)}$$

which are consistent, because $S(k)S(-k)=\mathbb{I}$, and imply Eq. (18). Equations (A2)–(A4) define a special *reflection-transmission* algebra $\mathcal{A}$, which has been introduced in a more general form in the study[43–46] of pointlike defects in integrable systems. Notice that, although k dependent, $S(k)$ is scale invariant.

Time reversal is realized in the algebra $\mathcal{A}$ by means of

$$Ta_i(k)T^* = -a_i(-k), \quad Ta_i^*(k)T^* = -a_i^*(-k). \quad \text{(A5)}$$

In fact, Eq. (A5) implies

$$T\varphi_{i,R}(t-x)T^* = -\varphi_{i,L}(-t+x), \quad \text{(A6)}$$

$$T\varphi_{i,L}(t+x)T^* = -\varphi_{i,R}(-t-x), \quad \text{(A7)}$$

which implement in turn the time-reversal transformation (9) and (10) on the solution (20) and (21).

For the construction of correlation functions we adopt the Fock representation of $\mathcal{A}$. We denote by Ω and $(\cdot,\cdot)$ the Fock vacuum state and the scalar product, using for the vacuum expectation values of the operators $\mathcal{O}_k$ the short notation

$$(\Omega, \mathcal{O}_1 \cdots \mathcal{O}_n \Omega) = \langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle. \quad \text{(A8)}$$

Since $a_i(k)\Omega=0$, the basic correlators are

$$\langle a_i(p)a_j^*(q)\rangle = 2\pi[\delta_{ij}\delta(p-q) + S_{ij}(p)\delta(p+q)],$$
$$\langle a^{*i}(p)a_j(q)\rangle = 0, \quad \text{(A9)}$$

which imply

$$\langle \varphi_{i_1,R}(\xi_1)\varphi_{i_2,R}(\xi_2)\rangle = \langle \varphi_{i_1,L}(\xi_1)\varphi_{i_2,L}(\xi_2)\rangle = \delta_{i_1 i_2}u(\mu\xi_{12}), \quad \text{(A10)}$$

$$\langle \varphi_{i_1,L}(\xi_1)\varphi_{i_2,R}(\xi_2)\rangle = \mathbb{S}_{i_1 i_2}u(\mu\xi_{12}), \quad \text{(A11)}$$

$$\langle \varphi_{i_1,R}(\xi_1)\varphi_{i_2,L}(\xi_2)\rangle = \mathbb{S}^t_{i_1 i_2}u(\mu\xi_{12}), \quad \text{(A12)}$$

where $\xi_{12}=\xi_1-\xi_2$,

$$u(\xi) = -\frac{1}{\pi}\ln(\mathrm{i}\xi + \epsilon), \quad \epsilon > 0, \quad \text{(A13)}$$

and $\mu > 0$ is an infrared mass parameter.[47] The normalization constants z_i which occur in Eqs. (20) and (21) depend on μ in the following way:

$$z_i = (2\pi)^{-1/2}\mu^{[(\sigma^2+\tau^2)+2\sigma\tau\mathbb{S}_{ii}]/2}\eta_i, \quad \text{(A14)}$$

where η_i are the *anyon* Klein factors needed to ensure the correct anyon exchange relations on different edges of the graph Γ. A simple representation is

$$\eta_i = :e^{\pi\mathrm{i}(\alpha_i+\alpha_i^*)}:, \quad \text{(A15)}$$

where $\{\alpha_i, \alpha_i : i=1,\ldots,n\}$ generate the auxiliary algebra

$$[\alpha_i, \alpha_j] = [\alpha_i^*, \alpha_j^*] = 0, \quad [\alpha_i, \alpha_j^*] = \mathrm{i}\frac{\kappa}{2}\epsilon_{ij}, \quad \text{(A16)}$$

with $\epsilon_{ij}=-1$ for $i<j$, $\epsilon_{ii}=0$, and $\epsilon_{ij}=1$ for $i>j$.

It is worth stressing that there is an action principle behind the whole structure (A1)–(A13). The action can be written in terms of the combinations

$$\varphi_i(t,x) = \tfrac{1}{2}[\varphi_{i,R}(t-x) + \varphi_{i,L}(t+x)], \quad \text{(A17)}$$

$$\widetilde{\varphi}_i(t,x) = \tfrac{1}{2}[\varphi_{i,R}(t-x) - \varphi_{i,L}(t+x)], \quad \text{(A18)}$$

and the auxiliary fields $\{\lambda_i(t,x), \widetilde{\lambda}_i(t,x)\}$ as follows. The bulk and boundary actions action are

$$\mathcal{S}_{\text{bulk}} = \int_{-\infty}^{+\infty} dt \int_0^{+\infty} dx \sum_{i=1}^n [\lambda_i(\partial_x\varphi_i + \partial_t\widetilde{\varphi}_i) + \widetilde{\lambda}_i(\partial_t\varphi_i + \partial_x\widetilde{\varphi}_i)] \times (t,x), \quad \text{(A19)}$$

$$\mathcal{S}_{\text{boundary}} = \frac{1}{2}\sum_{i=1}^{n}\int_{-\infty}^{+\infty} dt(\lambda_i\lambda_i - \widetilde{\lambda}_i\widetilde{\lambda}_i + \varphi_i\varphi_i + \widetilde{\varphi}_i\widetilde{\varphi}_i)(t,0) + \frac{1}{4}\sum_{i,j=1}^{n}\int_{-\infty}^{+\infty} dt[\widetilde{\varphi}_i(\mathbb{S}+\mathbb{S}^t)_{ij}\widetilde{\varphi}_j - \varphi_i(\mathbb{S}+\mathbb{S}^t)_{ij}\varphi_j - 2\varphi_i(\mathbb{S}-\mathbb{S}^t)_{ij}\widetilde{\varphi}_j](t,0), \qquad \text{(A20)}$$

respectively. The total action $\mathcal{S}=\mathcal{S}_{\text{bulk}}+\mathcal{S}_{\text{boundary}}$ is nondegenerate, with $\widetilde{\lambda}$ and λ being the conjugate momenta of φ and $\widetilde{\varphi}$, respectively. In agreement with Eq. (36), the only term breaking time-reversal invariance is the term proportional to $\mathbb{S}-\mathbb{S}^t$ in Eq. (A20). The bulk variation involves only $\mathcal{S}_{\text{bulk}}$. Varying with respect to λ and $\widetilde{\lambda}$, one gets the duality relations

$$\partial_t\widetilde{\varphi}(t,x,i) = -\partial_x\varphi(t,x,i), \qquad \text{(A21)}$$

$$\partial_x\widetilde{\varphi}(t,x,i) = -\partial_t\varphi(t,x,i). \qquad \text{(A22)}$$

The bulk variation with respect to φ and $\widetilde{\varphi}$ gives analogous relations between λ and $\widetilde{\lambda}$. The boundary variation involves both $\mathcal{S}_{\text{bulk}}$ and $\mathcal{S}_{\text{boundary}}$ and, as easily verified, generates the boundary condition (18).

A final comment concerns an interesting interplay between locality and time-reversal symmetry on Γ. A standard computation shows that at spacelike separated points $t_{12}^2 < x_{12}^2$,

$$[\varphi(t_1,x_1,i),\varphi(t_2,x_2,j)] = -[\widetilde{\varphi}(t_1,x_1,i),\widetilde{\varphi}(t_2,x_2,j)] = \frac{i}{4}(\mathbb{S}^t-\mathbb{S})_{ij}, \qquad \text{(A23)}$$

implying that the time-reversal breaking on Γ is accompanied by the violation of locality of φ and $\widetilde{\varphi}$.[48] One can easily check however that this violation does not affect the locality of the currents $j_{\pm}$, which belong to the observables of the theory. More about quantum field theory on graphs (also away from criticality) can be found in Refs. 18–20, 28, and 41–50, where some basic elements[51,52] of the spectral theory of differential operators on graphs ("quantum graphs") have been used.

APPENDIX B: CRITICAL $\mathbb{S}$ MATRICES FOR GENERIC n

First of all, the matrix R which rotates the vector $(0,0,\ldots,0,1)$ in $\mathbf{v}$ can be taken in the form

$$R_{ij} = \begin{cases} 0 & \text{if } i<j=1,2,\ldots,n-1 \\ \dfrac{-1}{\sqrt{(n-j)(n-j+1)}} & \text{if } i>j=1,\ldots,n-1 \\ \sqrt{\dfrac{n-i}{n-i+1}} & \text{if } i=j=1,\ldots,n-1 \\ \dfrac{1}{\sqrt{n}} & \text{if } i=1,\ldots,n, \quad j=n, \end{cases} \qquad \text{(B1)}$$

As well known, the matrix $\mathbb{S}' \in O(n-1)$ can be parametrized in terms of the $(n-1)(n-2)/2$ rotation matrices $\{r_{i,j}(\vartheta_{ij}): i,j=1,\ldots,n-1, i<j\}$ each of them rotating at the angle ϑ_{ij} in the ij plane. If $\det(\mathbb{S}')=1$ one has

$$\mathbb{S}' = \left(\prod_{i=n-2}^{1} r_{i,n-1}\right)\left(\prod_{i=n-3}^{1} r_{i,n-2}\right)\cdots(r_{2,3}r_{1,3})r_{1,2}. \qquad \text{(B2)}$$

The only delicate point is the domain of the generalized Euler angles ϑ_{ij}, which turns out to be[53]

$$\vartheta_{ij} \in \begin{cases} [-\pi,\pi) & \text{for } j=i+1 \\ [-\pi/2,\pi/2] & \text{for } j>i+1. \end{cases} \qquad \text{(B3)}$$

Finally, in the case $\det(\mathbb{S}')=-1$ one can simply multiply the right-hand side of Eq. (B3) by the matrix r, which *reflects*, for instance, along the first axis.

APPENDIX C: THE $\psi_1-\psi_2$ MIXING

The mixing between ψ_1 and ψ_2 is described by the two-point functions

$$\langle\psi_1^*(t_1,x_1,i_1)\psi_2(t_2,x_2,i_2)\rangle = z_{i_1}z_{i_2}[\mathcal{D}(vt_{12}-x_{12})]^{\sigma\tau\delta_{i_1i_2}}[\mathcal{D}(vt_{12}+x_{12})]^{\sigma\tau\delta_{i_1i_2}} \times[\mathcal{D}(vt_{12}-\widetilde{x}_{12})]^{\sigma^2\mathbb{S}^t_{i_1i_2}}[\mathcal{D}(vt_{12}+\widetilde{x}_{12})]^{\tau^2\mathbb{S}_{i_1i_2}}, \qquad \text{(C1)}$$

$$\langle\psi_2^*(t_1,x_1,i_1)\psi_1(t_2,x_2,i_2)\rangle = \text{(C1)}, \quad \text{with } \sigma\leftrightarrow\tau. \qquad \text{(C2)}$$

Combining Eqs. (C1) and (C2) with Eqs. (53) and (54), one finds that under a scaling transformation (55) a generic two-point function transforms according to

$$\langle\psi_{\alpha_1}^*(\varrho t_1,\varrho x_1,i_1)\psi_{\alpha_2}(\varrho t_2,\varrho x_2,i_2)\rangle = \varrho^{-\mathbb{D}_{\alpha_1\alpha_2,i_1i_2}}\langle\psi_{\alpha_1}^*(t_1,x_1,i_1)\psi_{\alpha_2}(t_2,x_2,i_2)\rangle, \qquad \text{(C3)}$$

where $\mathbb{D}$ is the $2n\times 2n$ matrix

$$\mathbb{D} = \begin{pmatrix} D & B \\ B^t & D \end{pmatrix}, \qquad \text{(C4)}$$

with D given by Eq. (57) and

$$B = 2\sigma\tau\mathbb{I}_n + \sigma^2\mathbb{S}^t + \tau^2\mathbb{S}. \qquad \text{(C5)}$$

The eigenvalues of the matrix $\mathbb{D}/4$ provide the dimensions capturing the $\psi_1-\psi_2$ mixing. We will prove now that n of the eigenvalues of $\mathbb{D}/4$ vanish and that the remaining n coincide precisely with the dimensions d_i given by Eq. (58). For this purpose we compute the characteristic polynomial $\det(\mathbb{D}-x\mathbb{I}_{2n})$. First we move to the basis in which $\mathbb{S}$ has the forms (45) and (46), performing the transformation

$$\begin{pmatrix} \mathbb{O} & 0 \\ 0 & \mathbb{O} \end{pmatrix}\begin{pmatrix} D & B \\ B^t & D \end{pmatrix}\begin{pmatrix} \mathbb{O}^t & 0 \\ 0 & \mathbb{O}^t \end{pmatrix} = \begin{pmatrix} D_{\text{d}} & B_{\text{bd}} \\ B_{\text{bd}}^t & D_{\text{d}} \end{pmatrix}. \qquad \text{(C6)}$$

In this basis D_{d} is diagonal, whereas B_{bd} is block diagonal. At this point we use the identity[54]

$$\det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det(M)\det(Q-PM^{-1}N), \qquad \text{(C7)}$$

where M, N, P, and Q are $n\times n$ blocks and M is invertible. Let us apply Eq. (C7) to $\det(\mathbb{D}-x\mathbb{I}_{2n})$ with $\mathbb{D}$ given by Eq.

(C6). For $D_d - x\mathbb{I}_n$ to be invertible we assume for the moment that $x \neq \sigma^2 + \tau^2 + 2\sigma\tau s_i$ with s_i defined by Eq. (59). One gets

$$\det(\mathbb{D} - x\mathbb{I}_{2n}) = \det(D_d - x\mathbb{I}_n) \times \det[D_d - x\mathbb{I}_n + B_{bd}(D_d - x\mathbb{I}_n)^{-1}B_{bd}^t]. \tag{C8}$$

Being determinants of diagonal and of block diagonal matrices, the two factors in the right-hand side of Eq. (C8) are easily computed. One finds

$$\det(D_d - x\mathbb{I}_n) = \prod_{i=1}^{n} (x - \sigma^2 - \tau^2 - 2\sigma\tau s_i), \tag{C9}$$

$$\det[D_d - x\mathbb{I}_n + B_{bd}(D_d - x\mathbb{I}_n)^{-1}B_{bd}^t] = \frac{\prod_{i=1}^{n} [x(x - 2\sigma^2 - 2\tau^2 - 4\sigma\tau s_i)]}{\prod_{i=1}^{n} (x - \sigma^2 - \tau^2 - 2\sigma\tau s_i)}. \tag{C10}$$

Notice that the factor (C9) cancels precisely the denominator of Eq. (C10). Therefore, the characteristic polynomial we are looking for is

$$\det(\mathbb{D} - x\mathbb{I}_{2n}) = \prod_{i=1}^{n} [x(x - 2\sigma^2 - 2\tau^2 - 4\sigma\tau s_i)], \tag{C11}$$

which extends for any x by continuity and proves our statement.

APPENDIX D: COMPOSITE TWO-FERMION OPERATORS

We examine here the stability of the critical points under the perturbation with the composite operators

$$\Phi_1(t,x,i) =: \psi_1^* \psi_2 :(t,x,i) \sim :e^{i\sqrt{\pi}\zeta_-[\varphi_{i,R}(vt-x) - \varphi_{i,L}(vt+x)]}:, \tag{D1}$$

$$\Phi_2(t,x,i) =: \psi_2^* \psi_1 :(t,x,i) \sim :e^{-i\sqrt{\pi}\zeta_-[\varphi_{i,R}(vt-x) - \varphi_{i,L}(vt+x)]}:. \tag{D2}$$

The relative two-point correlation functions are easily derived. One finds

$$\begin{aligned}\langle \Phi_1^*(t_1,x_1,i_1)\Phi_1(t_2,x_2,i_2)\rangle &= \langle \Phi_2^*(t_1,x_1,i_1)\Phi_2(t_2,x_2,i_2)\rangle \\ &\sim [\mathcal{D}(vt_{12} - x_{12})]^{\zeta_-^2 \delta_{i_1 i_2}} [\mathcal{D}(vt_{12} + x_{12})]^{\zeta_-^2 \delta_{i_1 i_2}} \\ &\quad \times [\mathcal{D}(vt_{12} - \tilde{x}_{12})]^{-\zeta_-^2 \mathbb{S}_{i_1 i_2}^t} [\mathcal{D}(vt_{12} + \tilde{x}_{12})]^{-\zeta_-^2 \mathbb{S}_{i_1 i_2}},\end{aligned} \tag{D3}$$

$$\begin{aligned}\langle \Phi_1^*(t_1,x_1,i_1)\Phi_2(t_2,x_2,i_2)\rangle &= \langle \Phi_2^*(t_1,x_1,i_1)\Phi_1(t_2,x_2,i_2)\rangle \\ &\sim [\mathcal{D}(vt_{12} - x_{12})]^{-\zeta_-^2 \delta_{i_1 i_2}} [\mathcal{D}(vt_{12} + x_{12})]^{-\zeta_-^2 \delta_{i_1 i_2}} \\ &\quad \times [\mathcal{D}(vt_{12} - \tilde{x}_{12})]^{\zeta_-^2 \mathbb{S}_{i_1 i_2}^t} [\mathcal{D}(vt_{12} + \tilde{x}_{12})]^{\zeta_-^2 \mathbb{S}_{i_1 i_2}}.\end{aligned} \tag{D4}$$

As before, the response of Eqs. (D3) and (D4) under the scaling transformation (55) defines the $2n \times 2n$ matrix

$$\tilde{\mathbb{D}} = \zeta_-^2 \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \otimes (\mathbb{S} + \mathbb{S}^t - 2\mathbb{I}), \tag{D5}$$

with the dimensions of the operators (D1) and (D2) being the eigenvalues of $\tilde{\mathbb{D}}/2$. One easily finds that n of these eigenvalues vanish. The remaining n are given by

$$\tilde{d}_i = \zeta_-^2 (1 - s_i), \tag{D6}$$

where s_i are defined by Eq. (59). Subtracting from Eq. (D6) the dimensions of the same operators on the line, one finds the nontrivial boundary dimensions

$$\tilde{d}_i^{(\text{boundary})} = -\zeta_-^2 s_i. \tag{D7}$$

At this point one can repeat the analysis performed in Sec. VII for perturbations with a single-fermion operator. Comparing Eqs. (62) and (D7) and using $\zeta_-^2 > 0$, we see that in the attractive regime $g_+ < g_-$ the stability properties of the critical points under the two different perturbations are the same. In the repulsive case $g_+ > g_-$ the behavior is inverted. The directions which were stable become unstable under perturbations with Eqs. (D1) and (D2) and vice versa.

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[33] The usually used parameter K in our notation coincides for $\kappa=1$ with $\zeta_-^2=\zeta_+^{-2}$.
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[37] The parameter g_c, adopted in Ref. 22, is related to ϑ by $\sin\vartheta=\pm 2\sqrt{3}g_c/(3+g_c^2)$, with the two different signs corresponding to the families $\chi_\pm$.
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